# A METHOD OF SIMULATING SOME HEAT CONDUCTION PROBLEMS WITH MOVING BOUNDARIES

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A method of simulating some problems of the heat conduction type with moving boundaries is examined. Two problems with exact solutions are simulated.

In a number of heat conduction (or diffusion) problems it is necessary to investigate regions where a phase interface varies with time. The law of motion of the interface must be found from physical considerations. Examples are the well-known problems of Stefan, Verigin, and others. There are no analytical solutions of such problems, with rare exceptions, and the known numerical methods require great effort and time. The electrical modeling method [1–4] is very effective in giving an approximate solution to problems of this type.

The present paper describes a method of simulating certain problems with moving boundaries, problems of the Stefan and Verigin type, the method being different from that of [1–4].

We shall examine the equations

$$\bar{c}(x)\frac{\partial \bar{U}}{\partial t} = \frac{\partial}{\partial x}\left(\bar{k}(x)\frac{\partial \bar{U}}{\partial x}\right) - \bar{b}(x)\bar{U}(x, t),$$

$$0 < t < T, \quad 0 < x < y(t), \tag{1}$$

$$c(x)\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial U}{\partial x}\right) - b(x)U(x, t),$$

$$0 < t < T, \; y(t) < x < l \tag{2}$$

with the initial condition

$$\bar{U}(x, 0) = \bar{\varphi}(x), \; 0 < x < c, \; U(x, 0) = \varphi(x),$$

$$c < x < l, \; c = y(0) \tag{3}$$

and boundary conditions

$$\bar{k}\bar{U}_x(0, t) = -\bar{q}(t), \quad kU_x(l, t) = -q(t) \quad (\bar{q}, q \geqslant 0), \tag{4}$$

where

$$\bar{k} = \bar{k}(0), \; k = k(l).$$

The law of motion of the phase interface is given by the equation

$$\gamma(y(t), t)\, y'(t) = k(y(t))\, U_x(y(t), t) -$$

$$- \bar{k}(y(t))\, \bar{U}_x(y(t), t) + \Phi(y(t), t). \tag{5}$$

On the phase interface one of the following conditions is given:

$$U(y(t), t) = \bar{U}(y(t), t) = U_0 = \text{const} \tag{6}$$

(Stefan's problem) or

$$U(y(t), t) = \bar{U}(y(t), t), \tag{7}$$

$$\bar{\alpha}\,\bar{U}_x(y(t), t) = \alpha\, U_x(y(t), t) \tag{8}$$

(Verigin's problem). The presence of condition (5) reduces these problems to a number of nonlinear ones [5].

We shall assume that the following congruence conditions are fulfilled:

$$\bar{\varphi}'(0) = -\bar{q}(0), \; \varphi'(l) = -q(0), \; \bar{\varphi}(c) = \varphi(c), \; \bar{\alpha}\bar{\varphi}'(c) = \alpha\varphi'(c),$$

and that a unique solution exists for both problems.

**Implicit difference scheme for approximate solution of the problem.** We introduce a network in the regions $[0, y(t)]$, $[y(t), l]$, by dividing the segment $0 \leq x \leq l$ at points $x_i$ into N equal parts with step h, where $x_i = ih$, $i = 0, \ldots, N$, in such a way that the point c is one of the division points. Let $x_k = kh = c$. We choose the steps with respect to time, $\tau_k$, to depend on h in such a way that in each interval $\tau_k$ the value of y changes by h, i.e., $y_k - y_{k-1} = h$, where $y_k$ is the approximate value of y(t) at time $t = t_k = \sum_{i=1}^{k} \tau_i$.

We replace problem (1)–(8) by the following difference problem, in order to determine $\tau_n$ and approximate values $\bar{U}_{in}$, $U_{in}$ of functions $\bar{U}(x, t)$ and $U(x, t)$ at the points $x_i$, $t_n$:

$$\bar{c}_i \delta_t \bar{U}_{in} = \frac{1}{h}\left(\bar{k}_i' \delta_x \bar{U}_{in} - \bar{k}_i'' \delta_x \bar{U}_{i-1,\,n}\right) - \bar{b}_i \bar{U}_{in},$$

$$i = 1, 2, \ldots, k + n - 1, \tag{9}$$

$$c_i \delta_t U_{in} = \frac{1}{h}\left(k_i' \delta_x U_{in} - k_i'' \delta_x U_{i-1,\,n}\right) - b_i U_{in},$$

$$i = k + n - 1, \ldots, N - 1, \; n = 1, \ldots, N \tag{10}$$

$$\bar{k}\,\delta_x \bar{U}_{0n} = -\bar{q}(t_{n-1}) = -\bar{q}_{n-1}, \tag{11}$$

$$k\,\delta_x U_{N-1,\,n} = -q(t_{n-1}) = -q_{n-1}, \tag{12}$$

$$\bar{U}_{i0} = \bar{\varphi}_i, \; i = 1, \ldots, k; \; U_{i0} = \varphi_i, \; i = k, \ldots, N. \tag{13}$$

At points of the interface y(t) we have

$$\bar{U}_{k+n,\,n} = U_{k+n,\,n} = U_0 \tag{14}$$

(Stefan's problem);

$$\bar{U}_{k+n,\,n} = U_{k+n,\,n}, \tag{15}$$

$$\bar{\alpha}\,\delta_x \bar{U}_{k+n-1,\,n} = \alpha \delta_x U_{k+n,\,n} \tag{16}$$

(Verigin's problem);

$$\gamma_{n-1} h/\tau_n = k_n \delta_x U_{k+n,\,n} - \bar{k}_n \delta_x \bar{U}_{k+n-1}, \; k_n = k(y_n). \tag{17}$$

Here

$$\delta_t z_{in} = \frac{1}{\tau_n}(z_{in} - z_{i,\,n-1}),\quad \delta_x z_{in} = \frac{1}{h}(z_{i+1,\,n} - z_{in}),$$

$$z_i' = z_{i+1/2},\quad z_i'' = z_{i-1/2}.$$

We solve the nonlinear system of equations obtained for $U_{in}$, $\overline{U}_{in}$, $\tau_n$ with the aid of iteration. Thus, if $U_{ik}$, $\overline{U}_{ik}$, $\tau_k$ (k = 1, . . . , n − 1) have already been found, then to determine $U_{in}$, $\overline{U}_{in}$, $\tau_n$, we iterate according to the scheme

$$\overline{c}_i\delta_t\overline{U}_{in}^{(s)} = \frac{1}{h}(\overline{k}_i'\,\delta_x\overline{U}_{in}^{(s)} - \overline{k}_i''\,\delta_x\overline{U}_{i-1,\,n}^{(s)}) - \overline{b}_i\overline{U}_{in}^{(s)}, \quad (18)$$

$$c_i\,\delta_t U_{in}^{(s)} = \frac{1}{h}(k_i'\delta_x U_{in}^{(s)} - k_i''\,\delta_x U_{i-1,\,n}^{(s)}) - b_i U_{in}^{(s)}, \quad (19)$$

$$\overline{U}_{i0}^{(s)} = \overline{\varphi}_i,\ U_{i0}^{(s)} = \varphi_i, \quad (20)$$

$$\overline{k}\,\delta_x\overline{U}_{0n}^{(s)} = -\overline{q}_{n-1},\quad k\,\delta_x U_{N-1,\,n}^{(s)} = -q_{n-1}, \quad (21)$$

$$\overline{U}_{k+n,\,n}^{(s)} = U_{k+n,\,n}^{(s)} = U_0, \quad (22)$$

$$\overline{U}_{k+n,\,n}^{(s)} = U_{k+n,\,n}^{(s)}, \quad (23)$$

$$\overline{\alpha}\,\delta_x\overline{U}_{k+n-1,\,n}^{(s)} = \alpha\delta_x U_{k+n,\,n}^{(s)}, \quad (24)$$

$$\tau_n^{(s)} = \frac{h}{F_{n-1}}\left\{\gamma_{n-1} + \frac{\tau_n^{(s-1)}}{h}\left[\overline{q}_{n-1} - q_{n-1} + \overline{k}_n\,\delta_x\overline{U}_{k+n-1,\,n}^{(s)} - k_n\,\delta_x U_{k+n,\,n}^{(s)}\right]\right\}, \quad (25)$$

where

$$\delta_t z_{in}^{(s)} = \frac{1}{\tau_n^{(s)}}(z_{in}^{(s)} - z_{i,\,n-1}),\quad F_{n-1} = \overline{q}_{n-1} - q_{n-1} + \Phi_{n-1}.$$

If the given value is $\tau_n^{(0)} > 0$, then, from the system (18)–(24) with s = 0 we find $\overline{U}_{in}^{(0)}$, $U_{in}^{(0)}$, and from (25) we obtain $\tau_n^{(1)}$, and so on.

Assuming that $\overline{q}(t'') \geqslant \overline{q}(t')$, $q(t'') \geqslant q(t')$ for any $t'' \geq t' \geq 0$, $\overline{q}(t) \geqslant q(t)$, $-\overline{q}(0) - \Phi_0 = \overline{k}_i'\delta_x\overline{\varphi}_i = -q(0)$, $k_i'\delta_x\varphi_i - k_i''\delta_x\varphi_{i-1} \geqslant 0$, $\overline{k}_i'\delta_x\overline{\varphi}_i - \overline{k}_i''\delta_x\overline{\varphi}_{i-1} \geqslant 0$ when $0 \leq x \leq l$, and using the method of [5, 6], it may be shown that, for any assigned value $\tau_n^{(0)} > 0$, the iterations (18)–(25) will be uniquely determined for any s ≥ 0, and $\overline{U}_{in}^{(s)}$, $U_{in}^{(s)}$, $\tau_n^{(s)}$ will converge to the solution $\overline{U}_{in}$, $U_{in}$, $\tau_n$ of the system (9)–(16).

The above results may be extended to the more general case. Specifically, we shall examine the equations

$$\overline{c}(x, t, \overline{U}, \overline{U}_x, \overline{U}_{xx})\,\frac{\partial\overline{U}}{\partial t} = \frac{\partial}{\partial t}\left(\overline{k}(x)\,\frac{\partial\overline{U}}{\partial t}\right) - \overline{b}(x, t, \overline{U}, \overline{U}_x, \overline{U}_{xx})\,\overline{U}(x, t),$$

$$c(x, t, U, U_x, U_{xx})\,\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial U}{\partial x}\right) - b(x, t, U, U_x, U_{xx})\,U(x, t)$$

with initial conditions (3), boundary conditions

$$\overline{k}\overline{U}_x(0, t) = -\overline{q}(t, \overline{U}(0, t)),\ kU_x(l, t) = -q(t, U(l, t)),$$

conditions at the moving interface (6)–(8) and

$$\gamma(x, t, U, \overline{U}, U_x, \overline{U}_x)\,y'(t) = k(x)\,U_x(x, t) - \overline{k}(x)\,\overline{U}_x(x, t) + \Phi(x, t, U, \overline{U}, U_x, \overline{U}_x).$$

The last expression is examined with x = y(t). By putting

$$\overline{q}_{n-1} = \overline{q}(t_{n-1}, \overline{U}(0, t_{n-1})),\ q_{n-1} = q(t_{n-1}, U(l, t_{n-1})),$$

$$\overline{c}_i = \overline{c}_{i,\,n-1} = \overline{c}(x, t, \overline{U}, \overline{U}_x, \overline{U}_{xx})_{t=t_{n-1}},$$

$$c_i = c_{i,\,n-1} = c(x, t, U, U_x, U_{xx})_{t=t_{n-1}},$$

$$\overline{b}_i = \overline{b}_{i,\,n-1} = \overline{b}(x, t, \overline{U}, \overline{U}_x, \overline{U}_{xx})_{t=t_{n-1}},$$

$$b_i = b_{i,\,n-1} = b(x, t, U, U_x, U_{xx})_{t=t_{n-1}}$$

$$\Phi_{i,\,n-1} = \Phi(x, t, U, \overline{U}, U_x, \overline{U}_x)_{t=t_{n-1}}$$

in (9)–(25), it can be shown that the iterations converge to the solution of the system (9)–(17).

As regards agreement between the difference problem solution and the exact solution, the reader should refer to [5, 6].

**Electrical circuit of model.** The system (18)–(25) and its corresponding iteration are not very suitable for numerical solution because of the large number of calculations, but the system does permit a simple solution on the electrical resistance model examined below.

Let us examine the electrical circuit shown in the figure. We shall denote the potentials at the nodes $P_0, \ldots, P_N$ by $V_0, \ldots, V_N$, respectively. Let

$$\overline{R}_i = \frac{h^2}{\overline{k}_i''}R,\quad \overline{\overline{R}}_i = \frac{\tau_n}{\overline{c}_i}R,$$

$$\overline{R}_i' = \frac{1}{\overline{b}_i}R\quad (i = 1, \ldots, k+n-1),$$

$$R_i = \frac{h^2}{k_i''}R,\quad \overline{R}_i = \frac{\tau_n}{c_i}R,\quad R_i' = \frac{1}{b_i}R\quad (i = k+n+1, \ldots, N-1),$$

$$\overline{R}_{k+n} = 0,\ R_\Gamma = rR_N,\ \overline{R}_\Gamma = rR_1,$$

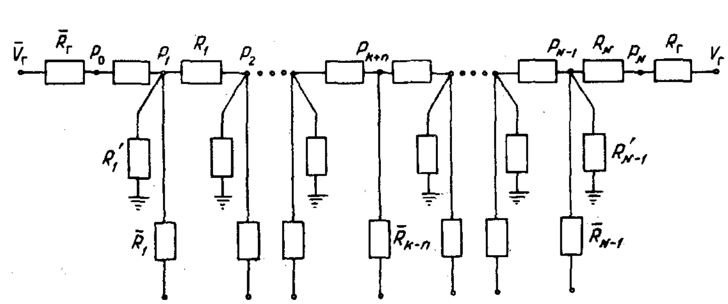
$$\overline{V}_\Gamma = rh\overline{q}_{n-1}V,\ V_\Gamma = -rhq_{n-1}V,\ V_{k+n} = U_0 V,$$

where R, V are scale factors for resistance and the variable U, and r is a sufficiently large positive number. At the free ends of the resistors $\overline{R}_i$ we assign potentials $V_{i,\,n-1} = U_{i,\,n-1}V$, where $V_{i0} = \varphi_i V$.

Using Kirchhoff's law for each nodal point $P_i$, it is easy to show that the system of equations for potentials $V_{in}$ at nodes $P_i$ coincides with the finite-difference system of the Stefan problem, if only $\tau_n$ is known.

For the nonlinear Stefan problem

$$\overline{R}_i = \frac{h^2}{\overline{k}_i''}R,\quad \overline{\overline{R}}_i = \frac{\tau_n}{\overline{c}_{i,\,n-1}}R,\quad \overline{R}_i' =$$



Electrical circuit of model.

$$=\frac{1}{b_{i,\,n-1}}\,R,\ i=1,\ \ldots,\ k+n-1,$$

$$R_i=\frac{h^2}{k_i''}\,R,\ \ \bar R_i=\frac{\tau_n}{c_{i,\,n-1}}\,R,\ \ R_i'=$$

$$=\frac{1}{b_{i,\,n-1}}\,R,\ i=k+n+1,\ \ldots,\ N-1,$$

$$\bar R_{k+n}=0,\ \ R_\Gamma=rR_N,\ \ \bar R_\Gamma=rR_1,$$

$$\bar V_\Gamma=rh\bar q_{n-1}V,\ V_\Gamma=-rhq_{n-1}V,\ V_{k+n}=U_0V.$$

To model the Verigin problem we must put

$$R_i=\frac{h^2}{\bar k_i''}\,R,\ \ \bar R_i=\frac{\tau_n}{\bar c_i}\,R,$$

$$R_i'=\frac{1}{\bar b_i}\,R,\ \ i=1,\ \ldots,\ k+n-1,$$

$$R_i=m\,\frac{h^2}{k_i''}\,R,\ \ \bar R_i=m\,\frac{\tau_n}{c_i}\,R,$$

$$R_i'=\frac{m}{b_i}\,R,\ \ i=k+n+1,\ldots,\ N-1,$$

$$\bar R_{k+n}=\infty,\ \ \bar R_\Gamma=rR_1,\ \ R_\Gamma=rR_N,$$

$$\bar V_\Gamma=rh\bar q_{n-1}V,\ \ V_\Gamma=-mrhq_{n-1}V,$$

where $m = k_n\bar\alpha/\bar k_n\alpha$.

In the nonlinear case we have

$$R_i=\frac{h^2}{\bar k_i}\,R,\ \ \bar R_i=\frac{\tau_n}{\bar c_{i,\,n-1}}\,R,\ \ R_i'=\frac{1}{\bar b_{i,\,n-1}}\,R,$$

$$R_i=m\,\frac{h^2}{k_i''}\,R,\ \ \bar R_i=m\,\frac{\tau_n}{c_{i,\,n-1}}\,R,\ \ R_i'=\frac{m}{b_{i,\,n-1}}\,R.$$

$$\bar R_{k+n}=\ ,\ \ \bar R_\Gamma=rR_1,\ \ R_\Gamma=rR_N,$$

$$\bar V_\Gamma=rh\bar q_{n-1}V,\ \ V_\Gamma=-mrhq_{n-1}V.$$

In conformity with the iteration process in (18)–(25), we assign arbitrary $\tau_n^{(0)}>0$. Then the resistances $\bar R_i$ are determined from the calculation formulas, and thus all the $\bar U_{in}^{(0)},\ U_{in}^{(0)}$ in the electrical circuit are found. Then $\tau_n^{(1)}$ is calculated from (25), and $\bar U_{in}^{(1)},\ U_{in}^{(1)}$ are found, and so on. The iteration is stopped when a given accuracy for $\tau_n$ is attained. In going from $n = n_1$ to $n = n_1 + 1$, the point of phase separation $P_{k+n}$ moves to the next node to the right. The changes which must then be introduced into the electrical circuit in the vicinity of node $P_{k+n}$ are simple and self-evident.

As an illustration of the application of the above method, we present the results of modeling two simple problems and compare them with the exact solutions.

Let us examine the following Stefan problem:

$$\bar U_t=4\bar U_{xx}\{0<x<3,\ 0<t<23.4\},\ U\equiv 0,\ \bar U(0,x)=0$$

with boundary conditions

$$\bar k\bar U_x(0,t)=-0.4555\,\frac{1}{\sqrt t},\ \ kU_x(3,t)=0.$$

Its analytic solution has the form

$$\bar U(x,t)=1-\frac{\operatorname{erf}(x/\sqrt{2t})}{\operatorname{erf}(\alpha/\sqrt 2)},$$

$$U\equiv 0,\ \ y(t)=\alpha\sqrt t,\ \ \alpha=0.620.$$

In the model we took $h = 0.2$; the result found by modeling was $\alpha = 0.628$.

Let us now examine the Verigin problem.

$$\bar U_t=2\bar U_{xx}\{0<x<y(t)\},\ U_t=U_{xx}\{y(t)<x<2\},$$

$$-2\bar U_x(0,t)=0.7740\,\frac{1}{\sqrt t},$$

$$-U_x(2,t)=0.7283\,\frac{1}{\sqrt t}\,\exp\left(-\frac{1}{t}\right),$$

$$U(x,0)=0,\ \bar U_x(y(t),t)=U_x(y(t),t),\ \bar U(y(t),t)=U(y(t),t),$$

$$y'(t)=U_x(y(t),t)-2\bar U_x(y(t),t).$$

Its analytic solution has the form

$$\bar U(x,t)=1-1.3723\operatorname{erf}(x/\sqrt{16t}),$$

$$U(x,t)=1.0344[1-\operatorname{erf}(x/\sqrt{8t})],$$

$$y(t)=\alpha\sqrt t,\ \ \alpha=0.724.$$

In the model we took $h = 0.1$, and the result found by modeling was $\alpha = 0.726$. The error in determining $\alpha$ was 1.5% for the Stefan problem and 0.3% for the Verigin problem. Both problems were modeled on an EI-12 electro-integrator. On the average 4–5 iterations were required to determined $\tau_n$, taking into account that $\tau_n^{(0)}=\tau_{n-1}$. The number of iterations may be reduced if $\tau_n^{(0)}$ is chosen close to $\tau_n$.

It should be noted that the principal error of the method is determined by the quantity h.

NOTATION

$U(x,t)$, $\bar U(x,t)$—body temperature or pressure; $t$—time; $x$—space variable; $q(t)$—heat or fluid flux; $k$, $\bar k$—thermal conductivities or filtration coefficients; $\gamma$—latent heat of fusion (or solidification), or specific porosity; $c$, $\bar c$—specific heat; $b$, $\bar b$—heat transfer or diffusion coefficients with respect to external medium; $U_0$—fusion (solidification) temperature.

REFERENCES

1. D. R. Otis, Transactions of ASME, 79, 759–764, 1957.
2. L. A. Kozdoba and V. I. Makhnenko, IFZh, no. 4, 1964.
3. I. F. Zherebyat'ev and A. T. Luk'yanov, IFZh, no. 6, 1964.
4. V. T. Chernoval, Proceedings of the 4-th Inter-University Conference on the Applications of Physical and Mathematical Simulation in Various Branches of Engineering [in Russian], collection 1, Izd. MEI, 1962.
5. B. M. Budak, F. P. Vasil'ev, and A. B. Uspenskii, Numerical Methods in Gas Dynamics [in Russian], vol. 4, Izd. MGU, 1965.
6. F. P. Vasil'ev, DAN SSSR, 157, no. 6, 1964.
7. L. I. Kamynin, ZhVM i MF, 2, no. 5, 1962.
8. N. I. Verigin, DAN SSSR, OTN, no. 5, 1952.

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